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$$\frac{[(m-x)\cos\theta-(n-y)\sin\theta]^2}{a^2}+\frac{[(m-x)\sin\theta+(n-y)\cos\theta]^2}{b^2}=1....(1).$$

The equation to ellipse, center A, is

$$x^2/a^2+y^2/b^2=1....(2).$$

The invariant of these equations are

$$\Delta=\Delta_1=-a^4b^4.$$
$$\Theta=a^2b^2[b^2(m^2+n^2)+c(m\sin\theta+n\cos\theta)^2-c^2\sin^2\theta-3a^2b^2].$$
$$\Theta_1=a^2b^2[b^2m^2+a^2n^2-c^2\sin^2\theta-3a^2b^2), \text{ where } c=a^2-b^2.$$

Let $m=r\cos\varphi$, $n=r\sin\varphi$.

$$\therefore\ \Theta=a^2b^2\{r^2[a^2\sin^2(\theta+\varphi)+b^2\cos^2(\theta+\varphi)]-c^2\sin^2\theta-3a^2b^2\}$$
$$=a^2b^2(r^2A-c^2\sin^2\theta-3a^2b^2), \text{ suppose.}$$

$$\Theta_1=a^2b^2[r^2(a^2\sin^2\varphi+b^2\cos^2\varphi)-c^2\sin^2\theta-3a^2b^2]$$
$$=a^2b^2(r^2B-c^2\sin^2\theta-3a^2b^2), \text{ suppose.}$$

The condition that the ellipses should touch is

$$(\Theta\Theta_1-9\Delta\Delta_1)^2=4(\Theta^2-3\Delta\Theta_1)(\Theta_1^2-3\Delta_1\Theta).$$

The locus of B is

$$A^2B^2r^8-2(A+B)[(5a^2b^2+c^2\sin^2\theta)AB-2a^2b^2(A^2+B^2)]r^6$$
$$+[(A^2+B^2)(c^4\sin^4\theta-6a^2b^2c^2\sin^2\theta-27a^4b^4)+2AB(2c^4\sin^4\theta$$
$$+12a^2b^2c^2\sin^2\theta+27a^4b^4)]r^4-2c^4\sin^4\theta(A+B)(3a^2b^2+c^2\sin^2\theta)r^2$$
$$+c^6\sin^6\theta(4a^2b^2+c^2\sin^2\theta)=0.$$

This equation represents two loci, one when the moving ellipse is tangent externally, the other when the moving ellipse is tangent internally.

If a is the average area of the former, the chance is $p=\pi ab/a$.

121. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of the pentagon formed by joining five random points on the surface of a given circle.

Solution by the PROPOSER.

Let a=radius of given circle, A=its area, Δ=required average, Δ'=average area when the five points are taken on both the circle A and a concentric annulus B, Δ_1=average area when four points are taken on A and one on B.

Then $(\triangle'+\triangle)A=5B(\triangle_1-\triangle)$. But $\triangle:\triangle'=A:A+B$.

$\therefore\ \triangle'=(A+B)\triangle/A.\quad\therefore\ \triangle=\frac{5}{6}\triangle_1$.

Let one point O be on the circumference of the circle and the other four points P, Q, R, S anywhere on its surface. Let $OP=u$, $OS=x$, $OR=y$, $OQ=z$, $\angle AOE=\theta$, $\angle BOE=\varphi$, $\angle COE=\psi$, $\angle DOE=\delta$. Then $OA=2a\sin\theta=u'$, $OB=2a\sin\varphi=x'$, $OC=2a\sin\psi=y'$, $OD=2a\sin\delta=z'$.

Area $OPQRS=\frac{1}{2}uz\sin(\theta-\delta)+\frac{1}{2}yz\sin(\delta-\psi)+\frac{1}{2}xy\sin(\psi-\varphi)=C$.

$$\therefore\ \triangle=\frac{5}{6}\triangle_1=\frac{5\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}Cuxyzd\theta d\delta d\psi d\varphi dudxdydz}{6\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}uxyzd\theta d\delta d\psi d\varphi dudxdydz}$$

$$=\frac{20}{\pi^4a^8}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}Cuxyzd\theta d\delta d\psi d\varphi dudxdydz$$

$$=\frac{20}{3\pi^4a^6}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}[4au\sin(\theta-\delta)\sin\delta+4ay\sin(\delta-\psi)\sin\delta$$
$$+3xy\sin(\psi-\varphi)]uxy\sin^2\delta d\theta d\delta d\psi d\varphi dudxdy$$

$$=\frac{160}{9\pi^4a^3}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}[3u\sin(\theta-\delta)\sin\delta+3x\sin(\psi-\varphi)\sin\psi$$
$$+4a\sin(\delta-\psi)\sin\delta\sin\psi]ux\sin^2\delta\sin^2\psi d\theta d\delta d\psi d\varphi dudx$$

$$=\frac{320}{9\pi^4a}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}[3u\sin(\theta-\delta)\sin\delta+4a\sin(\psi-\varphi)\sin\psi\sin\varphi$$
$$+4a\sin(\delta-\psi)\sin\delta\sin\psi]u\sin^2\delta\sin^2\psi\sin^2\varphi d\theta d\delta d\psi d\varphi du$$

$$=\frac{2560a^2}{9\pi^4}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}[\sin(\theta-\delta)\sin\theta\sin\delta+\sin(\delta-\psi)\sin\delta\sin\psi$$
$$+\sin(\psi-\varphi)\sin\psi\sin\varphi]\sin^2\theta\sin^2\delta\sin^2\psi\sin^2\varphi d\theta d\delta d\psi d\varphi$$

$$=\frac{320a^2}{9\pi^4}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}[4\sin(\theta-\delta)\sin\theta\sin\delta(\psi-\sin\psi\cos\psi)+3\sin^2\psi$$
$$+4\sin(\delta-\psi)\sin\delta\sin\psi(\psi-\sin\psi\cos\psi)-\sin^4\psi-3\psi\sin\psi\cos\psi]\sin^2\theta\sin^2\delta\sin^2\psi d\theta d\delta d\psi$$

$$=\frac{10a^2}{27\pi^4}\int_0^{\pi}\int_0^{\theta}[105\delta-72\delta^2\sin\delta\cos\delta+84\delta\sin^2\delta-120\delta\sin^4\delta-105\sin\delta\cos\delta$$

$$-82\sin^3\delta\cos\delta+32\sin^5\delta\cos\delta+96\sin(\theta-\delta)\sin\theta\sin\delta(\delta-\sin\delta\cos\delta)^2]\sin^2\theta\sin^2\delta d\theta d\delta$$

$$=\frac{5a^2}{162\pi^4}\int_0^{\pi}(360\theta^2-144\theta^3\sin\theta\cos\theta-360\theta^2\sin^4\theta+252\theta^2\sin^2\theta+336\theta\sin^5\theta\cos\theta$$

$$-132\theta\sin^3\cos\theta-615\theta\sin\theta\cos\theta+72\sin^8\theta-250\sin^6\theta-301\sin^4\theta+255\sin^2\theta)\sin^2\theta d\theta$$

$$=\frac{5a^2}{12\pi}\left(5-\frac{1001}{144\pi^2}\right).$$

MISCELLANEOUS.

116. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Prove that

$$-\left|\,a^2\beta^2\gamma^2\,\right|^2\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2 = \begin{vmatrix} \begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} \begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} \begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} \begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} \begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & c & \gamma_1 \\ b & d & e & \gamma_2 \\ c & e & f & \gamma_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} \begin{vmatrix} a & b & c & \gamma_1 \\ b & d & e & \gamma_2 \\ c & e & f & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} \begin{vmatrix} a & b & c & \gamma_1 \\ a & b & c & \gamma_2 \\ c & e & f & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \end{vmatrix}$$

[From Muir's *Determinants*.]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x, y, z be the minors with respect to α_1, α_2, α_3 in the first row, u, v, w the minors with respect to β_1, β_2, β_3 in the second row, and r, s, t the minors with respect to γ_1, γ_2, γ_3 in the third row of the determinant $\triangle$ on the right hand side of the equality. Then

$$=-\begin{vmatrix} \alpha_1x-\alpha_2y+\alpha_3z, & \beta_1x-\beta_2y+\beta_3z, & \gamma_1x-\gamma_2y+\gamma_3z \\ \alpha_1u-\alpha_2v+\alpha_3w, & \beta_1u-\beta_2v+\beta_3w, & \gamma_1u-\gamma_2v+\gamma_3w \\ \alpha_1r-\alpha_2s+\alpha_3t, & \beta_1r-\beta_2s+\beta_3t, & \gamma_1r-\gamma_2s+\gamma_3t \end{vmatrix}$$

$$=\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_2 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}\begin{vmatrix} x & y & z \\ u & v & w \\ r & r & t \end{vmatrix}=(\alpha_1\beta_2\gamma_3)\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}$$